Problem 5
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Let the vertices of the connection be numbered as shown in the figure. Let the potential of the kth vertex be u_k, the current flowing from the kth vertex towards the 0th vertex i_k, while the current flowing from the kth vertex to the $(k+1)$th one (modulo 4, of course) i_{k+4} (positive is the current when flowing in the branch in the mentioned direction). The following differential equations between the currents and voltages for specific circuit elements are well-known:

(1) $u_1 - u_2 = L_1 \frac{\mathrm{d}i_5}{\mathrm{d}t}$

(2) $u_2 - u_3 = L_1 \frac{\mathrm{d}i_6}{\mathrm{d}t}$

(3) $u_3 - u_4 = L_2 \frac{\mathrm{d}i_7}{\mathrm{d}t}$

(4) $u_4 - u_1 = L_2 \frac{\mathrm{d}i_8}{\mathrm{d}t}$

(5) $i_1 = C_1 \frac{\mathrm{d}(u_1 - u_0)}{\mathrm{d}t}$

(6) $i_2 = C_1 \frac{\mathrm{d}(u_2 - u_0)}{\mathrm{d}t}$

(7) $i_3 = C_2 \frac{\mathrm{d}(u_3 - u_0)}{\mathrm{d}t}$

(8) $i_4 = C_2 \frac{\mathrm{d}(u_4 - u_0)}{\mathrm{d}t}$.

From the first law of Kirchhoff the following equations can be got:

(9) $i_1 = i_8 - i_5$

(10) $i_2 = i_5 - i_6$

(11) $i_3 = i_6 - i_7$

(12) $i_4 = i_7 - i_8$.

It is easy to see, that this system is linear, so it has definite eigenfrequencies. In order to find them, we assume that all quantities depend on time as $x = Xe^{i\omega t}$, where ω denotes the angular eigenfrequency (they always occur in pairs $(\omega, -\omega)$, this can be considered as phase difference). Using this substitution in Eqs 1–12, and simplifying with $e^{i\omega t}$ gives the following equations:

(1) $U_1 - U_2 = i\omega L_1 I_5$

(2) $U_2 - U_3 = i\omega L_1 I_6$

(3) $U_3 - U_4 = i\omega L_2 I_7$

(4) $U_4 - U_1 = i\omega L_2 I_8$

(5) $I_1 = i\omega C_1 (U_1 - U_0)$

(6) $I_2 = i\omega C_1 (U_2 - U_0)$

(7) $I_3 = i\omega C_2 (U_3 - U_0)$

(8) $I_4 = i\omega C_2 (U_4 - U_0)$

(9) $I_1 = I_8 - I_5$

(10) $I_2 = I_5 - I_6$

(11) $I_3 = I_6 - I_7$

(12) $I_4 = I_7 - I_8$.

From Eqs 5–8 the values of $U_i - U_0$, consequently $U_i - U_{i+1}$ can be determined in terms of I_i, which can be equated with the values from Eqs 1–4. For example,

$$U_1 - U_2 = (U_1 - U_0) - (U_2 - U_0) = \frac{I_1}{i\omega C_1} - \frac{I_2}{i\omega C_1} = i\omega L_1 I_5$$

$$\frac{I_8 - I_5}{i\omega C_1} - \frac{I_5 - I_6}{i\omega C_1} = i\omega L_1 I_5$$

$$(13) \quad -\frac{2}{C_1 L_1} I_5 + \frac{1}{C_1 L_1} I_6 + \frac{1}{C_1 L_1} I_8 = -\omega^2 I_5.$$

The same calculation for other I_ks gives

$$(14) \quad -\left(\frac{1}{C_1 L_1} + \frac{1}{C_2 L_1}\right) I_6 + \frac{1}{C_1 L_1} I_5 + \frac{1}{C_2 L_1} I_7 = -\omega^2 I_6$$

$$(15)\quad -\frac{2}{C_2L_2}I_7+\frac{1}{C_2L_2}I_6+\frac{1}{C_2L_2}I_8=-\omega^2I_7$$

$$(16)\quad -\left(\frac{1}{C_1L_2}+\frac{1}{C_2L_2}\right)I_8+\frac{1}{C_1L_2}I_5+\frac{1}{C_2L_2}I_7=-\omega^2I_8.$$

The eqs 13–16 can be written together in the matrix form

$$\begin{pmatrix} -\frac{2}{C_1L_1} & \frac{1}{C_1L_1} & 0 & \frac{1}{C_1L_1} \\ \frac{1}{C_1L_1} & -\left(\frac{1}{C_1L_1}+\frac{1}{C_2L_1}\right) & \frac{1}{C_2L_1} & 0 \\ 0 & \frac{1}{C_2L_2} & -\frac{2}{C_2L_2} & \frac{1}{C_2L_2} \\ \frac{1}{C_1L_2} & 0 & \frac{1}{C_2L_2} & -\left(\frac{1}{C_1L_2}+\frac{1}{C_2L_2}\right) \end{pmatrix}\begin{pmatrix} I_5 \\ I_6 \\ I_7 \\ I_8 \end{pmatrix}=-\omega^2\begin{pmatrix} I_5 \\ I_6 \\ I_7 \\ I_8 \end{pmatrix}.$$

From this form it is easy to see, that the problem is to find the eigenvalues of the matrix on the left hand side: these eigenvalues give the values of $-\omega^2$. The characteristic equation of the matrix is (with the help of Maple):

$$\lambda^4+\left(\frac{1}{L_1C_1}+\frac{1}{L_2C_1}+\frac{1}{L_1C_2}+\frac{1}{L_2C_2}\right)\lambda^3+\left(\frac{1}{L_1^2C_1^2}+\frac{1}{L_2^2C_2^2}+\frac{2}{L_1L_2C_1^2}+\frac{2}{L_1L_2C_2^2}+\frac{2}{L_1^2C_1C_2}+\right.$$

$$\left.\frac{2}{L_2^2C_1C_2}+\frac{10}{L_1L_2C_1C_2}\right)\lambda^2+\left(\frac{4}{L_1^2L_2C_1^2C_2}+\frac{4}{L_1^2L_2C_1C_2^2}+\frac{4}{L_1L_2^2C_1^2C_2}+\frac{4}{L_1L_2^2C_1C_2^2}\right)\lambda=0.$$

From the equation it is obvious that $\lambda=0$, consequently $\omega=0$ is a solution: knowing this, we can divide the equation by λ; in the following, we consider it this way. Now, using the fact that $L_2\gg L_1$ and $C_2\gg C_1$, we can ignore the terms containing $\frac{1}{L_2}$ and $\frac{1}{C_2}$. This way, the equation becomes

$$\lambda^3+\frac{3}{L_1C_1}\lambda^2+\frac{1}{L_1^2C_1^2}\lambda=0.$$

0 is obviously the solution of the equation, however, this comes from ignoring terms; the other two eigenvalues can be got from the quadratic equation $\lambda^2+\frac{3}{L_1C_1}\lambda+\frac{1}{L_1^2C_1^2}=0$, the roots of which are $\lambda_1=-\frac{3+\sqrt{5}}{2}\frac{1}{L_1C_1}$ and $\lambda_2=-\frac{3-\sqrt{5}}{2}\frac{1}{L_1C_1}$. Now we have to determine the third root of the equation. From the previous it can be found, that this λ is ignorable compared to $\frac{1}{L_1C_1}$; this means that higher-order terms can be ignored compared to the two least-order one; the equation then becomes (ignoring the terms ignorable compared to the greatest in each coefficient):

$$\frac{1}{L_1^2C_1^2}\lambda+\frac{4}{L_1^2L_2C_1^2C_2}=0$$

$$\lambda_3=-\frac{4}{L_2C_2}.$$

Now we have all eigenvalues of the matrix of the system; now we're going to calculate the eigenfrequencies from $\lambda=-\omega^2$.

$\lambda_1=-\frac{3+\sqrt{5}}{2}\frac{1}{L_1C_1}$; from this, $\omega_1=\sqrt{\frac{3+\sqrt{5}}{2}\frac{1}{L_1C_1}}=\frac{1+\sqrt{5}}{2}\frac{1}{\sqrt{L_1C_1}}$. From $\lambda_2=-\frac{3-\sqrt{5}}{2}\frac{1}{L_1C_1}$, $\omega_2=\sqrt{\frac{3-\sqrt{5}}{2}\frac{1}{L_1C_1}}=\frac{1-\sqrt{5}}{2}\frac{1}{\sqrt{L_1C_1}}$ comes. Eigenvalue $\lambda_3=-\frac{4}{L_2C_2}$ gives the eigenfrequency $\omega_3=\sqrt{\frac{4}{L_2C_2}}=\frac{2}{\sqrt{L_2C_2}}$. We investigate now the case of the fourth eigenfrequency, $\lambda_4=0$: in this case there is a nonzero eigenvector $(1,1,1,1)^*$, meaning that a constant current flows round in the outer ring, and there is no current in the 0th vertex: this is physically able; there is no voltage inducating in the coils, so vertices 1–4 are on the same potential u_1, while u_0 is independent of this. Thus the angular eigenfrequencies of the circuit are $\omega_1=\frac{1+\sqrt{5}}{2}\frac{1}{\sqrt{L_1C_1}}$, $\omega_2=\frac{1-\sqrt{5}}{2}\frac{1}{\sqrt{L_1C_1}}$, $\omega_3=\frac{2}{\sqrt{L_2C_2}}$ and $\omega_4=0$.