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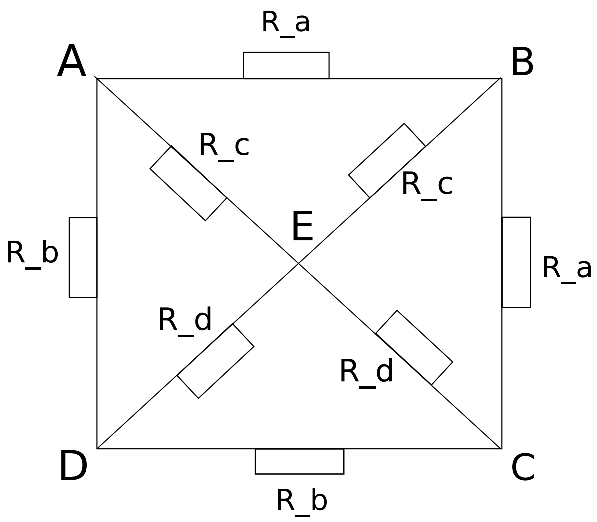
IPhO Physics cup 2012

P(r)o(bl)em 5

Let's write the components of this circuit as resistors with complex resistance. Like this:

$$R_a = 2\pi f i L_1$$
$$R_b = 2\pi f i L_2$$
$$R_c = \frac{1}{2\pi f i C_1}$$
$$R_d = \frac{1}{2\pi f i C_2}$$



Now, we can work with this circuit using Kirchhoff's laws.

First, we need to determine, how many natural frequencies will there be. We can use the fact that this number is the same as the number of degrees of freedom of this system, because a system of N coupled oscillators has N degrees of freedom (these describe the oscillators, while the couplings are described by the oscillators' states).

We need 4 loop currents (4 degrees of freedom) to describe this system. We can view the circuit as a pyramid, for which, if we know the loop currents of 4 faces, we can determine the current of the 5th, and since every loop is drawn along the pyramid's edges, it's made by joining some of the faces together (so its current is a superposition of these faces' currents). These 4 (independent) linear equations, along with 4 (independent) equations for 1st Kirchhoff law, allow us to determine all the currents flowing through edges. So there will be 4 natural frequencies.

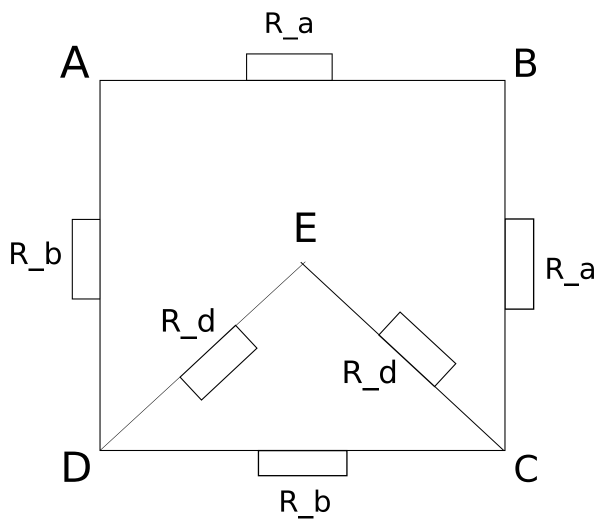
One of them is $f = 0$ (like, a DC, passing through a loop of inductors, while there are constant charges on the capacitors, constant magnetic fluxes so no voltage is induced across the inductors... yes, this is the plural to flux, there's no shorter form like fluci or whatever... and everything is constant).

In such situations, it's wise to expect the frequencies to be a low and a high one. Let's investigate these 2 situations separately.

Let's first assume the frequency f is small enough for

$$|R_c| \gg \max\left(|R_a|, |R_b|, |R_d|\right)$$

so that the current flowing through the R_c components is negligible in comparison to all the other currents in the circuit. In such situation, we can cross out the R_c components from the circuit, and get this:



We can now imagine that one of the nodes is actually composed of 2 nodes, connected with a perfectly conducting wire (which is basically a power source with 0 potential difference) - we can also imagine that we cut the circuit at a node and connect the cut points with a wire. So, we have a source (the added wire) and a load connected to it, whose impedance is (when cutting point A, for example; the result doesn't depend on our choice of the point to cut, since all so-made circuits are equivalent)

$$Z = 2R_a + R_b + \frac{1}{\frac{1}{R_b} + \frac{2}{R_d}} = 4\pi fiL_1 + 2\pi fiL_2 + \frac{1}{\frac{1}{2\pi fiL_2} + \frac{2\pi fiC_2}{2}} \approx 2\pi fiL_2 + \frac{2\pi fiL_2}{1 - 2\pi^2 f^2 L_2 C_2}$$

At natural frequency, impedance of the load is 0 - the 2 points connected with the wire must have the same potentials and how much current flows out of one of them into the load, that much flows into the other one from the load (and that much flows through the wire). If the current is non-zero (which must be, or there is no current flowing anywhere in the circuit), we need $Z = 0$ to get 0 potential difference. From this, we obtain (all operations were equivalent)

$$1 = \pi^2 f^2 L_2 C_2$$

$$f = \frac{1}{\pi\sqrt{L_2 C_2}}$$

Originally, we used the assumption of large R_c, so we need to prove it now. From the strong inequalities in the problem description, we already know that $|R_a| \ll |R_b|$, $|R_c| \gg |R_d|$. So we only need to show that

$$|R_c| \gg |R_b|$$

we do a few (equivalent) operations

$$4\pi^2 f^2 L_2 C_1 \ll 1$$

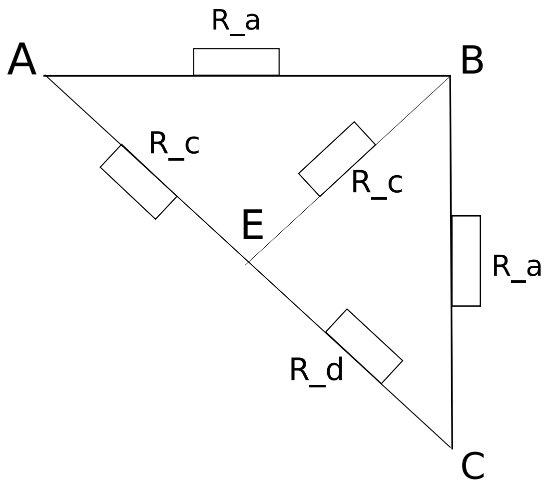
$$\frac{4C_1}{C_2} \ll 1$$

which is true (4 is a constant close to 1, so it does not affect a strong inequality). So we found one of the frequencies.

Now, we assume that the frequency is large enough, so that

$$|R_b| \gg \max(|R_a|, |R_c|, |R_d|)$$

and therefore, we can cross out the 2 L_2 inductors, and get a circuit (the left C_2 is not part of any circuit, so we ignore it)



Its impedance is (here, for cutting at point C, for example)

$$Z = R_a + R_d + \frac{1}{\frac{1}{R_a+R_c} + \frac{1}{R_c}} = 2\pi f i L_1 + \frac{1}{2\pi f i C_2} + \frac{1}{\frac{2\pi f i C_1}{1-4\pi^2 f^2 L_1 C_1} + 2\pi f i C_1}$$

which must be, for the same reasons, zero, from which we get

$$4\pi^2 f^2 C_1 L_1 - \frac{C_1}{C_2} = \frac{1}{\frac{1}{1-4\pi^2 f^2 L_1 C_1} + 1} = \frac{1-4\pi^2 f^2 L_1 C_1}{2-4\pi^2 f^2 L_1 C_1}$$

and using the $C_1 \ll C_2$ inequality (which, in the end, allows us to neglect $\frac{C_1}{C_2}$ from this equation's leftmost side), we get

$$1 - 12\pi^2 f^2 C_1 L_1 + 16\pi^4 f^4 L_1^2 C_1^2 = 0$$

$$f^2 = \frac{3 \pm \sqrt{5}}{8\pi^2 L_1 C_1} = \frac{1+\phi_{1,2}}{4\pi^2 L_1 C_1}$$

where ϕ_1 is the golden cut and ϕ_2 its conjugate).

Here, we get the last 2 solutions:

$$f = \frac{\sqrt{1+\phi_1}}{2\pi\sqrt{L_1 C_1}}$$

$$f = \frac{\sqrt{1+\phi_2}}{2\pi\sqrt{L_1 C_1}}$$

Once again, we need to show that $|R_b| \gg |R_c|$ in both cases. We can also write it as

$$4\pi^2 f^2 L_2 C_1 \gg 1$$

$$\frac{(3+\sqrt{5})L_2}{2L_1} = 2.6\frac{L_2}{L_1} \gg 1$$
$$\frac{(3-\sqrt{5})L_2}{2L_1} = 0.38\frac{L_2}{L_1} \gg 1$$

which, once again, follows from the inequality $L_2 \gg L_1$ and the fact that multiplying it by a constant close to 1 doesn't affect its validity.

It's also possible to derive the general solution by changing 2 of the triangles in the circuit (I think the best is to do it with L_1 - C_1 - C_1 and L_2 - C_2 - C_2 triangles) to equivalent stars, which yields a much, much simpler to compute (the 2 rules for series and parallel circuits are sufficient, no need for diffs) circuit. I'm NOT going to try this.

P.S. A reverse way to determine all the loop currents from just 1st Kirchhoff law and 4 independent edge currents is also possible.

Instead of loop currents, we can also use node potentials / potential differences. Since we're not interested in exact values of node potentials (they depend on our choice of reference zero potential), we can choose node E to be 0 potential, and only need to know potentials of the remaining four (which are independent), relative to it. In other words, we only need 4 independent potential differences (and using 2nd Kirchhoff law, can determine the rest), which are our degrees of freedom.