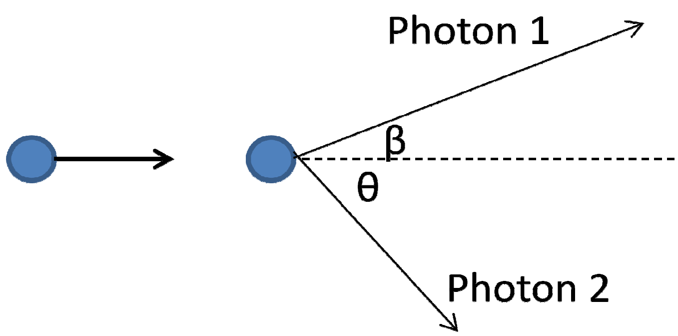
Problem No 9



Assume that after the electron hits the positron, photon 1 deflects β angle and photon 2 deflects θ angle from the electron's original path. This way, $\beta + \theta = \alpha$.

The electron gains the energy $\Delta E = e \cdot U = km_0c^2$. Therefore, its total energy is $E = (k+1)m_0c^2$.

And, it has the momentum (using energy-momentum relation):

$$E^2 = m_0{}^2c^4 + p^2c^2$$
$$(k+1)^2 m_0{}^2c^4 = m_0{}^2c^4 + p^2c^2$$
$$p^2c^2 = m_0{}^2c^4(k^2+2k)$$
$$p = m_0c\sqrt{k^2+2k}$$

When the electron hits the positron, the momentum and energy is conserved. The energy and the momentum of photon 1 and photon 2 are $E_1 = hf_1$ and $E_2 = hf_2$, and $p_1 = \frac{hf_1}{c}$ and $p_2 = \frac{hf_2}{c}$, respectively.

The equation of the conservation of momentum of the axis perpendicular to the original path of the electron:

$$p = p$$
$$0 = \frac{hf_1}{c}\sin\beta - \frac{hf_2}{c}\sin\theta$$
$$f_1 \sin\beta = f_2 \sin\theta \qquad (1)$$

The equation of the conservation of momentum of the axis parallel to the original path of the electron:

$$p = p$$
$$m_0c\sqrt{k^2+2k} = \frac{hf_1}{c}\cos\beta + \frac{hf_2}{c}\cos\theta$$
$$m_0c^2\sqrt{k^2+2k} = hf_1\cos\beta + hf_2\cos\theta \quad (2)$$

The equation of the conservation of the energy:

$$E = E$$

$$(k+1)m_0c^2 + m_0c^2 = hf_1 + hf_2$$

$$m_0c^2(k+2) = h(f_1 + f_2) \qquad (3)$$

Substituting (3) into (2), and substitute (1) afterwards:

$$(f_1 + f_2)\sqrt{k^2+2k} = (f_1 \cos\beta + f_2 \cos\theta)(k+2)$$

$$f_2(\frac{\sin\theta}{\sin\beta} + 1)\sqrt{k^2+2k} = (f_2 \frac{\sin\theta}{\sin\beta}\cos\beta + f_2\cos\theta)(k+2)$$

$$(\sin\theta + \sin\beta)\sqrt{k^2+2k} = (\sin\theta\cos\beta + \cos\theta\sin\beta)(k+2)$$

$$(\sin\theta + \sin\beta)\sqrt{k^2+2k} = \sin(\theta+\beta)(k+2)$$

Using the relation $\beta + \theta = \alpha$, we can get:

$$(\sin\theta + \sin\beta)\sqrt{k^2+2k} = \sin(\theta+\beta)(k+2)$$

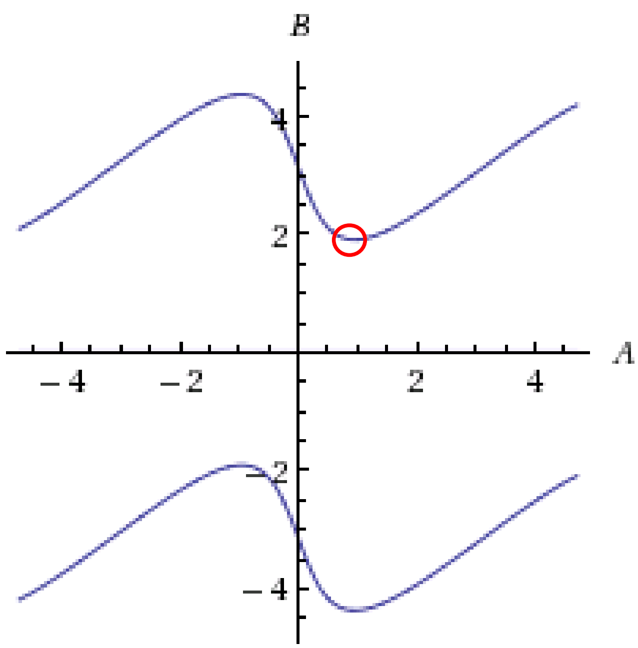
$$(\sin(\alpha-\beta) + \sin\beta)\sqrt{k^2+2k} = (k+2)\sin\alpha \qquad (4)$$

If we differentiate implicitly with respect to β:

$$(\sin(\alpha-\beta) + \sin\beta)\sqrt{k^2+2k} = (k+2)\sin\alpha$$

$$\left[\cos(\alpha-\beta)\left(\frac{d\alpha}{d\beta} - 1\right) + \cos\beta\right]\sqrt{k^2+2k} = (k+2)\cos\alpha\frac{d\alpha}{d\beta}$$

The implicit graph of the equation (4) >> using WolframAlpha. In this graph, α=B and β=A, and k=1.



The smallest possible angle of α will be the angle that $\frac{d\alpha}{d\beta}$ = 0 (look at the red circle on the graph).

$$\left[\cos(\alpha-\beta)\left(\frac{d\alpha}{d\beta}-1\right)+\cos\beta\right]\sqrt{k^2+2k}=(k+2)\cos\alpha\frac{d\alpha}{d\beta}$$

$$\left[-\cos(\alpha-\beta)+\cos\beta\right]\sqrt{k^2+2k}=0$$

$$\cos(\alpha-\beta)=\cos\beta$$

The solutions for the above equations are $\alpha=0$, $\alpha=2\beta$, etc. But if we examine the graph, the only correct solution is $\alpha=2\beta$.

For $\alpha=2\beta$, :

$$(\sin(\alpha-\beta)+\sin\beta)\sqrt{k^2+2k}=(k+2)\sin\alpha$$

$$(\sin\frac{\alpha}{2}+\sin\frac{\alpha}{2})\sqrt{k^2+2k}=(k+2)\sin\alpha$$

$$2\sin\frac{\alpha}{2}\sqrt{k^2+2k}=2(k+2)\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$

$$2\sin\frac{\alpha}{2}\left[\sqrt{k^2+2k}-\cos\frac{\alpha}{2}(k+2)\right]=0$$

Thus:

$$\sqrt{k^2+2k}-\cos\frac{\alpha}{2}(k+2)=0$$

$$\cos\frac{\alpha}{2}=\frac{\sqrt{k^2+2k}}{(k+2)}$$

$$\alpha=2\cos^{-1}\left[\frac{\sqrt{k^2+2k}}{(k+2)}\right]$$

For k=1:

$$\alpha=2\cos^{-1}\left[\frac{\sqrt{k^2+2k}}{(k+2)}\right]=2\cos^{-1}\left[\frac{\sqrt{3}}{3}\right]=1.91\text{ rad}$$

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