I will first introduce the concept of magnetic circuits, an ohm's law equivalent for magnetism.

First let's make an analogy with electric circuits:

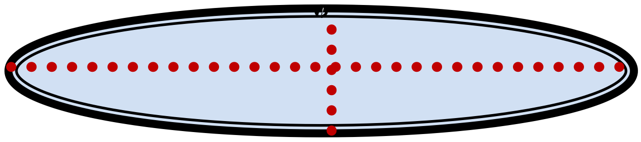
$$\boldsymbol{E} = -\boldsymbol{\nabla}\mathrm{V}$$

$$so, let \quad \boldsymbol{H} = \boldsymbol{\nabla}A$$

$$where\ A\ is\ the\ \text{magnetic}\ scalar\ \text{potential}$$

For we to define "A", $\oint \boldsymbol{H} \cdot d\boldsymbol{l} = 0$, so we must define that a specific place/plane isn't part of the system .

In this problem, we may choose this place as the plane inside of the circular loop:



Now we can define

$$\boldsymbol{H} = \boldsymbol{\nabla}A$$

Let's continue the analogy.

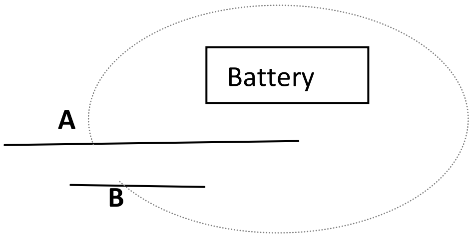
$$V_{AB=} \int_A^B \boldsymbol{E} \cdot d\boldsymbol{l} \xrightarrow{yields} A_{AB=} \int_A^B \boldsymbol{H} \cdot d\boldsymbol{l}$$

$$\boldsymbol{J} = \sigma \boldsymbol{E} \xrightarrow{yields} \boldsymbol{B} = \mu\mu_0 \boldsymbol{H}$$

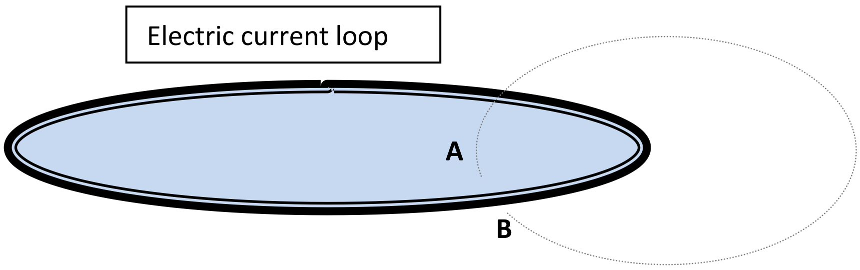
$$I = \iint \boldsymbol{J} \cdot d\boldsymbol{S} \xrightarrow{yields} \varphi = \iint \boldsymbol{B} \cdot d\boldsymbol{S}$$

$$V = IR \xrightarrow{yields} A = \varphi \mathbb{R} \ \ ; \mathbb{R} = reluctance$$

$$R = \frac{L}{\sigma S} \xrightarrow{yields} \mathbb{R} = \frac{L}{\mu\mu_0 S} \ \ ; S = surface\ area$$



$$V_{AB=} \int_A^B \boldsymbol{E} \cdot d\boldsymbol{l} = \varepsilon_{mf}$$



$$A_{AB} = \int_A^B \boldsymbol{H} \cdot d\boldsymbol{l} = I$$

If we have an inductor, with inductance L, we have: $I/\mathbb{R} = \varphi = L\,I \xrightarrow{yields} L = 1/\mathbb{R}$

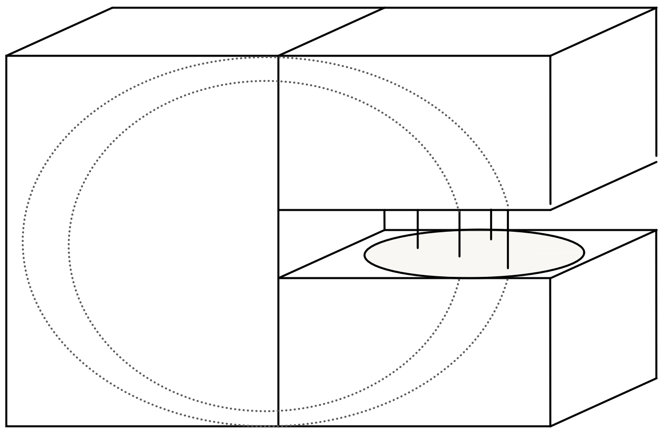
Now let´s solve the problem.

Let's separate the problem in 2 cases:

i) The diameter of the electric wire is approximately 'd' (L is small)
ii) The diameter of the electric wire is λ << d (L is big)

The first case:

We may assume that all the flux that goes through the current loop will go through the ferromagnetic, because, neglecting the reluctance of the ferromagnetic $\left(d \gg \frac{a}{\mu}\right)$, passing through the ferromagnetic , the field line will go through "d" in the air, against at least "π d" if it passed through the air near the wire, which has a bigger reluctance.



$$I = \varphi\mathbb{R} \qquad \varphi = \frac{I}{\mathbb{R}}$$

$$\mathbb{R} \approx \frac{4d}{\pi a^2 \mu_0}$$

$$\varphi = I\frac{\pi a^2 \mu_0}{4d}$$

[1] $$E_0 = \frac{\varphi I}{2} = I^2 \frac{\pi a^2 \mu_0}{8d}$$

The magnetic flux through the circular loop is constant, because it is made of a superconducting material. So the energy far away from the ferromagnetic is:

$$E_1 = \frac{LI^2}{2} = \frac{\varphi^2}{2L} = I^2 \frac{\pi a^2 \mu_0}{8d} \left(\frac{\pi a^2 \mu_0}{4dL} \right)$$

The work done is the difference of energy:

$$W = \Delta E = I^2 \left(\frac{\pi a^2 \mu_0}{8d} \right) \left(\frac{\pi a^2 \mu_0}{4dL} - 1 \right) \approx I^2 \left(\frac{\pi a^2 \mu_0}{8d} \right) \left(\frac{\pi a^2 \mu_0}{4dL} \right) = I^2 \left(\frac{\pi^2 a^4 {\mu_0}^2}{32 d^2 L} \right)$$

If we don't say use $\mu d \gg a$,

$$\mathbb{R} \approx \frac{4d}{\mu_0 \pi a^2} + \frac{8a}{a^2 \mu \mu_0} = \frac{8\pi a + 4d\mu}{a^2 \pi \mu \mu_0}$$

$$E_0 = \frac{\varphi I}{2} = \frac{I^2}{2} \frac{a^2 \pi \mu \mu_0}{8\pi a + 4d\mu}$$

$$E_1 = \frac{LI^2}{2} = \frac{\varphi^2}{2L} = \frac{I^2}{2} \frac{a^2 \pi \mu \mu_0}{8\pi a + 4d\mu} \left(\frac{a^2 \pi \mu \mu_0}{L(8\pi a + 4d\mu)} \right)$$

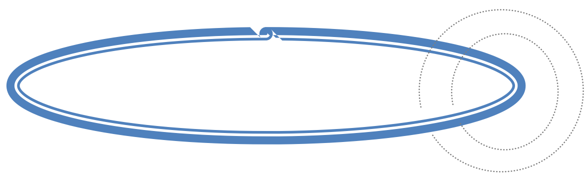
$$W = \Delta E = \frac{I^2}{2} \frac{a^2 \pi \mu \mu_0}{8\pi a + 4d\mu} \left(\frac{a^2 \pi \mu \mu_0}{L(8\pi a + 4d\mu)} - 1 \right)$$

The second case:

The diameter of the electric wire λ << d:

Now just a little part of the magnetic flux will go through the ferromagnetic, because the reluctance of this path $\mathbb{R} \approx \frac{4d}{\pi a^2 \mu_0} \gg \frac{1}{L}$, because d >>π λ

The most part of the magnetic flux will through a region really near the wire.



The reluctance of the path that doesn't pass through the ferromagnetic will be

$$\mathbb{R}_0 = 1/L$$

This is because most of the flux will pass in the region really near the wire, the region with the least reluctance, so the reluctance of this is small volume is ≈1/L

$$\mathbb{R} = \frac{1}{L} = \frac{\varphi}{I} \qquad \mathbb{R}_0 = \frac{\varphi_0}{I} \approx \frac{\varphi}{I} = \frac{1}{L}$$

So we will have a parallel association or reluctances, the reluctance of the path that don't pass through the ferromagnetic, and of the path that does pass.

The reluctance of the path that passes through the ferromagnetic won't change much because of this:

the surface area of this new path, will be

$$\frac{\pi a^2}{4} - S, where\ S\ is\ the\ surface\ area\ of\ this\ new\ path.$$

$$But\ S \sim \mathrm{d}^2 \ll \mathrm{a}^2, so\ the\ surface\ doesn't change, and\ neither\ does\ the\ distance.$$

$$\text{The reluctance of one path is}: \frac{1}{L}, and\ the\ reluctance\ of\ the\ other\ path\ is\ \approx \frac{4d}{\pi a^2 \mu_0}$$

Let $\mathbb{R}_{equ\,i}$ is the equivalent reluctance of the system.

$$\frac{1}{\mathbb{R}_{equi}} = \frac{1}{\mathbb{R}_0} + \frac{1}{\mathbb{R}_1} = L + \frac{\pi a^2 \mu_0}{4d} = \frac{4Ld + \pi a^2 \mu_0}{4d}$$

$$\mathbb{R}_{equi} = \frac{4d}{4Ld + \pi a^2 \mu_0}$$

The flux will be:

$$\varphi = \frac{I}{\mathbb{R}} = I\left(\frac{4Ld + \pi a^2 \mu_0}{4d}\right)$$

The energy is:

$$E_0 = \frac{\varphi I}{2} = I^2\left(\frac{4Ld + \pi a^2 \mu_0}{8d}\right)$$

The energy far away from the ferromagnetic is:

$$E_1 = \frac{LI^2}{2} = \frac{\varphi^2}{2L} = I^2\left(\frac{4Ld + \pi a^2 \mu_0}{8d}\right)\left(\frac{4Ld + \pi a^2 \mu_0}{4Ld}\right)$$

The work done is the difference of energy:

$$W = \Delta E = I^2\left(\frac{4Ld + \pi a^2 \mu_0}{8d}\right)\left(\frac{4Ld + \pi a^2 \mu_0}{4Ld} - 1\right) = I^2\left(\frac{L}{2} + \frac{\pi a^2 \mu_0}{8d}\right)\left(\frac{\pi a^2 \mu_0}{4Ld}\right)$$
$$= I^2\left(\frac{\pi^2 a^4 \mu_0^2}{32Ld} + \frac{\pi a^2 \mu_0}{8d}\right)$$

And as we can see, the work is always positive! If $L \gg \frac{\pi a^2 \mu_0}{4}$,

$$W \approx I^2 \left(\frac{\pi a^2 \mu_0}{8d}\right)$$

If we don't use $\mu d \gg a$,

$$\mathbb{R}_1 \approx \frac{4d}{\mu_0 \pi a^2} + \frac{8a}{a^2 \mu \mu_0} = \frac{8\pi a + 4d\mu}{a^2 \pi \mu \mu_0}$$

$$\frac{1}{\mathbb{R}_{equi}} = \frac{1}{\mathbb{R}_0} + \frac{1}{\mathbb{R}_1} = L + \frac{a^2 \pi \mu \mu_0}{8\pi a + 4d\mu} = \frac{L(8\pi a + 4d\mu) + a^2 \pi \mu \mu_0}{8\pi a + 4d\mu}$$

$$\mathbb{R}_{equi} = \frac{8\pi a + 4d\mu}{L(8\pi a + 4d\mu) + a^2 \pi \mu \mu_0}$$

$$\varphi = \frac{I}{\mathbb{R}} = I\left(\frac{L(8\pi a + 4d\mu) + a^2 \pi \mu \mu_0}{8\pi a + 4d\mu}\right)$$

$$E_0 = \frac{\varphi I}{2} = \frac{I^2}{2}\left(\frac{L(8\pi a + 4d\mu) + a^2 \pi \mu \mu_0}{8\pi a + 4d\mu}\right) = \frac{I^2}{2}\left(\frac{L(8\pi a + 4d\mu) + a^2 \pi \mu \mu_0}{8\pi a + 4d\mu}\right)$$

$$E_1 = \frac{LI^2}{2} = \frac{\varphi^2}{2L} = \frac{I^2}{2}\left(\frac{L(8\pi a + 4d\mu) + a^2 \pi \mu \mu_0}{8\pi a + 4d\mu}\right)\left(1 + \frac{a^2 \pi \mu \mu_0}{L(8\pi a + 4d\mu)}\right)$$

$$\Delta E = \frac{I^2}{2}\left(\frac{L(8\pi a + 4d\mu) + a^2 \pi \mu \mu_0}{8\pi a + 4d\mu}\right)\left(\frac{a^2 \pi \mu \mu_0}{L(8\pi a + 4d\mu)}\right)$$

$$W = \Delta E = \frac{I^2}{2}\left(\frac{a^2 \pi \mu \mu_0}{(8\pi a + 4d\mu)} + \frac{a^4 \pi^2 \mu^2 {\mu_0}^2}{L(8\pi a + 4d\mu)^2}\right)$$

If $L \gg \frac{\pi a^2 \mu_0}{4}$,

$$W \approx \frac{I^2}{2}\left(\frac{a^2 \pi \mu \mu_0}{(8\pi a + 4d\mu)}\right)$$

Now let´s try the solutions with some numbers, knowing that the inductance of a circular loop is:

$$L = \mu_0 \frac{a}{2} \ln(\frac{4a}{\lambda} - 2 + \frac{1}{4})$$

If

$$\mu = 2000 , \quad a = 50cm, \quad d = 1cm, \quad \mu0 = 4\pi \times 10^{-7} \frac{\mathrm{V \cdot s}}{\mathrm{A \cdot m}}, \quad I = 1\,A$$

$$\lambda = 1 \text{ cm}$$

$$L = 1.88 \ \cdot 10^{-6} \ (IS)$$

$$W = \Delta E = \frac{I^2}{2}\frac{a^2\pi\mu\mu_0}{8\pi a + 4d\mu}\left(\frac{a^2\pi\mu\mu_0}{L(8\pi a + 4d\mu)} - 1\right) = 1.10 \cdot 10^{-4} J$$

$$W = \Delta E = \frac{I^2}{2}\left(\frac{a^2\pi\mu\mu_0}{(8\pi a + 4d\mu)} + \frac{a^4\pi^2\mu^2{\mu_0}^2}{L(8\pi a + 4d\mu)^2}\right) = 1.32 \cdot 10^{-4} J$$

A difference of 20%

Now with

$$\lambda = 0{,}0005 \text{ cm} = 0.005\text{mm} = 5\mu\text{m}$$

$$L = 3.8 * 10^{-6} \ (IS)$$

using

$$W = \Delta E = \frac{I^2}{2}\frac{a^2\pi\mu\mu_0}{8\pi a + 4d\mu}\left(\frac{a^2\pi\mu\mu_0}{L(8\pi a + 4d\mu)} - 1\right) = 4.9 \cdot 10^{-5} J$$

But using:

$$W = \Delta E = \frac{I^2}{2}\left(\frac{a^2\pi\mu\mu_0}{(8\pi a + 4d\mu)} + \frac{a^4\pi^2\mu^2{\mu_0}^2}{L(8\pi a + 4d\mu)^2}\right) = 7.0 \cdot 10^{-5} J$$

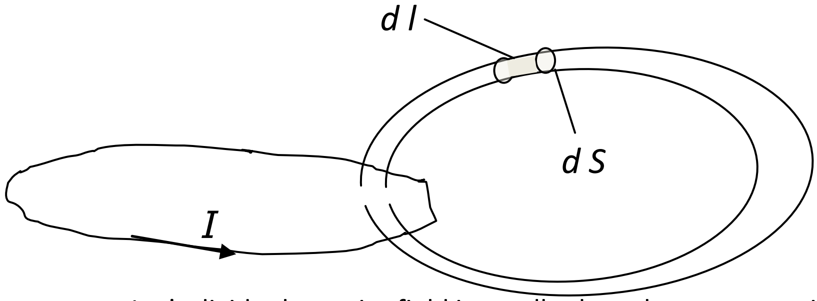
A difference of 42% from the first value.

As we can see, the smaller the diameter of the wire, the bigger the difference.

But this difference is big just for really small values of λ so it can be neglected almost all the times.

[1] The proof that the energy is equal to $\frac{\varphi \cdot I}{2}$ from the fact that the energy is $\int BH\, dV$ is below.

Let's consider the magnetic field of an arbitrary loop with current I .



Let's divide the entire field in small tube, whose generatrices are the field lines of B. In the figure is shown one of them. It's energy is BH/2 dl dS.

Let's find the energy in the volume of the entire tube. The flux $d\varphi = B\ dS$ through the tube cross section is constant along the tube, so it can be taken out of the integral:

$$dE = \frac{d\varphi}{2} \oint H\ dl = I\ \frac{d\varphi}{2}$$

Now, summing the energy of all elementary tubes:

$$E = \frac{1}{2} I \oint d\varphi = \frac{I\varphi}{2}$$

This formula can only be applies when the dependence of B vs H is linear, in other words, when $B = \mu\mu_0 H$, which is the case of this problem.

Bibliography:

'Basic laws of electromagnetism, I.E Irodov'

'Eletromagnetismo, William H. Hayt Jr'